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2008 J. Phys. A: Math. Theor. 41 015402

(http://iopscience.iop.org/1751-8121/41/1/015402)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 015402 (13pp) doi:10.1088/1751-8113/41/1/015402

Disk partition function and oscillatory rolling tachyons

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Received 6 August 2007, in final form 26 October 2007
Published 12 December 2007
Online at stacks.iop.org/JPhysA/41/015402

Abstract

An exact cubic open string field theory rolling tachyon solution was recently found by Kiermaier *et al* and Schnabl. This oscillatory solution has been argued to be related by a field redefinition to the simple exponential rolling tachyon deformation of boundary conformal theory. In the latter approach, the disk partition function takes a simple form. Out of curiosity, we compute the disk partition function for an oscillatory tachyon profile, and find that the result is nevertheless almost the same.

PACS numbers: 11.25.−w, 11.25.Uv, 11.25.Db, 11.25.Sq

1. Introduction

Recently there has been remarkable new analytic progress in the study of cubic open string field theory (OSFT) [1]. In particular, an exact rolling tachyon solution was found [2], related to the tachyon matter and decay of an unstable D-brane. The profile of the tachyon component of the full string field obtained by [2] from Witten's cubic OSFT is

$$T_\lambda(x^0) = \lambda\, \mathrm{e}^{\frac{1}{\sqrt{\alpha'}}x^0} + \sum_{n=2}^{\infty}(-1)^{n+1}\lambda^n\beta_n\, \mathrm{e}^{\frac{1}{\sqrt{\alpha'}}nx^0}, \tag{1}$$

where β_n are positive coefficients[3] with a known integral representation. The authors of [2] started from the exactly marginal operator

$$V = \mathrm{e}^{\frac{1}{\sqrt{\alpha'}}X^0}, \tag{2}$$

constructed the full OSFT solution recursively, adopting the gauge choice of [3], and obtained (1). Generalizations to superstrings have been reported in [4], and related work in [5].

[3] We follow the convention where the true minimum of the tachyon effective potential is at some $T > 0$ while keeping $\lambda > 0$. We work in units where $\alpha' = 1$.

The solution (1) has an oscillatory structure, as was suggested to be characteristic for the OSFT rolling tachyon by the previous investigations [6, 7]. On the other hand, in the boundary conformal field theory (BCFT) description of the same process[4], the tachyon field rolls monotonously, represented by the simple exponential (2). The apparent contradiction was addressed in [6]. The OSFT string field solution contains an infinite tower of other (massive) fields which are sourced by the rolling tachyon component. One can perform a field redefinition to boundary string field theory (BSFT) [9][5] variables, in such a way that all other fields except the tachyon are zero [11]. In the BSFT field coordinatization the tachyon can then turn out to be the simple exponential (2), while it was oscillatory in the OSFT frame [6]. Thus the marginal OSFT solution (where the tachyon component is off-shell) maps to a manifestly on-shell form. Further, it maps to the exactly marginal operator which gives a BCFT deformation. For the new full OSFT solution of [2] this was shown in [12]. Since the new rolling tachyon solution relates to the known BCFT deformation, in particular the time evolution of pressure of the associated tachyon matter has already been calculated in [13], it corresponds to the disk partition function of the BCFT with λV (2),

$$p(x^0) = Z_{\text{disk}}(x^0) = \frac{1}{1 + 2\pi\lambda\, \mathrm{e}^{x^0}}. \tag{3}$$

In this paper, we are reporting a curious observation. Suppose we were to consider BSFT with an oscillatory off-shell tachyon profile of the form (1). Consider the worldsheet CFT and turn on the boundary the tachyon field (1),

$$S = S_0 + \oint_{\partial\Sigma} \mathrm{d}t\, T_\lambda(X^0(t)), \tag{4}$$

it is off-shell and breaks the conformal invariance on the boundary. Suppose we attempt to do a straightforward calculation of the disk partition function, leaving the zero mode x^0 unintegrated. Given the oscillatory behavior of (1), we would probably expect the resulting disk partition function to be quite unwieldy and very different from (3).

However, when we perform string worldsheet theory analysis (along the lines of [13]), surprisingly we find that the result is *almost the same* as (3), with maximum 1% relative deviation. The deviation only appears at times close to the value $x^0 \sim -\ln 2\pi\lambda$. Apart from the deviation, there is no oscillatory behavior—at late times the disk partition functions become identical. We do not quite know how to interpret this curious observation. Apparently the field redefinitions involved in mapping from the oscillatory tachyon profile to the monotonously rolling one are not always so significant from the point of view of interesting observables. Further, while in our calculation the tachyon is of the form (1), the actual values of the coefficients β_n do not matter much—in particular, they (and the tachyon field) need not be the same as in [2]. Interpretational issues aside, we believe that the calculational tricks which we have used will be useful for other investigations and thus interesting in their own right.

2. The disk partition function

In the first quantized string worldsheet approach, we turn on the tachyon background (4). The disk partition function is (separating out the zero mode $X^0 = x^0 + X'^0$ and leaving it unintegrated)

$$Z_{\text{disk}}(x^0) = \int \mathcal{D}X'^0\, \mathcal{D}\vec{X}\, \mathrm{e}^{-S_0} \exp\left(-\oint_{\partial\Sigma} \mathrm{d}t\, T_\lambda(x^0 + X'^0(t))\right). \tag{5}$$

[4] For another reference on the relation between SFT solutions and deformations of BCFT, see [8].

[5] A pedagogical discussion of BSFT is also [10].

Note that, in the limit $\beta_{n>1} \to 0$, we expect to produce the familiar results for half S-brane [13].

By expanding in the boundary perturbation in (5) as a power series, and carefully following the calculational steps outlined in [14], the disk partition function is

$$
\begin{aligned}
Z_{\text{disk}}(x^0) &= \prod_{n=1}^{\infty}\sum_{N_n=0}^{\infty}\frac{\left((-1)^n\lambda^n\beta_n\,\mathrm{e}^{nx^0}\right)^{N_n}}{N_n!}\int \mathrm{d}t_1^{(n)}\cdots \mathrm{d}t_{N_n}^{(n)}\left\langle \prod_{n,i}\mathrm{e}^{nX'^0(t_i^{(n)})}\right\rangle \\
&= \sum_{\{N_1,N_2,\ldots\}=0}^{\infty}\left(\prod_{n=1}^{\infty}\frac{((-1)^n z_n)^{N_n}}{N_n!}\right)\cdot I(N_1,N_2,\ldots),
\end{aligned}
\tag{6}
$$

with

$$
z_n \equiv 2\pi\lambda^n\beta_n\,\mathrm{e}^{nx^0} > 0 \tag{7}
$$

and $\beta_1 = 1$, and where

$$
\begin{aligned}
I(N_1,N_2,\ldots) \equiv \int\left[\prod_{n=1}^{\infty}\prod_{i=1}^{N_n}\frac{\mathrm{d}t_i^{(n)}}{2\pi}\right]&\left[\prod_{n=1}^{\infty}\prod_{1\leqslant i<j\leqslant N_n}\left|\mathrm{e}^{it_i^{(n)}}-\mathrm{e}^{it_j^{(n)}}\right|^{2n^2}\right]\\
&\cdot\left[\prod_{1\leqslant n<m}^{\infty}\prod_{i=1}^{N_n}\prod_{j=1}^{N_m}\left|\mathrm{e}^{it_i^{(n)}}-\mathrm{e}^{it_j^{(m)}}\right|^{2nm}\right]
\end{aligned}
\tag{8}
$$

denotes an infinite product of coupled integrals.

The above formulae are just formal expressions before good domains of convergence are found. It is difficult to analyze the problem fully, so we will first study a simpler toy model.

3. A warm-up calculation: the Dyson series

We have two tasks at hand: (i) to try to calculate the integrals (8) and (ii) to try to control the series (6). These tasks appear to be rather challenging, so we will first consider a toy model calculation. It is reminiscent of the actual one but allows us to carry out both tasks.

We consider a series expansion, which we will call the 'Dyson series' from now on. It is inspired by the integration formula to compute the canonical partition function of a Dyson gas [15],

$$
\int\prod_{i=1}^{N}\frac{\mathrm{d}t_i}{2\pi}\left[\prod_{i<j}\left|\mathrm{e}^{\mathrm{i}t_i}-\mathrm{e}^{\mathrm{i}t_j}\right|^{\beta}\right] = \frac{\Gamma\left(1+\frac{\beta N}{2}\right)}{\left[\Gamma\left(1+\frac{\beta}{2}\right)\right]^N}, \tag{9}
$$

for which various proofs have been presented in the literature (see [16]). The integral (8) resembles an infinite product of decoupled Dyson gas integrals (9), except for the last cross coupling term in the square brackets in the integrand of (8). Let us first truncate the infinite product and keep just $n_{\max}$ first terms, with integer $n_{\max} \gg 1$. (In the end we will consider the limit $n_{\max} \to \infty$.) Then, consider the cross coupling term in the integrand of (8), which renders the integral difficult to evaluate. Let us rewrite it as

$$
\prod_{1\leqslant n<m}^{n_{\max}}\prod_{i=1}^{N_n}\prod_{j=1}^{N_m}\left|\mathrm{e}^{it_i^{(n)}}-\mathrm{e}^{it_j^{(m)}}\right|^{2nm} = \prod_{1\leqslant n<m}^{n_{\max}}\prod_{i=1}^{N_n}\prod_{j=1}^{N_m}\left(1-\frac{\mathrm{e}^{it_j^{(m)}}}{\mathrm{e}^{it_i^{(n)}}}\right)^{nm}\left(1-\frac{\mathrm{e}^{it_i^{(n)}}}{\mathrm{e}^{it_j^{(m)}}}\right)^{nm}. \tag{10}
$$

Now it turns out that the integral simplifies drastically if we replace the exponent nm in the first term on the rhs by n^2, and the second exponent nm by m^2. This step is clearly ad hoc.

However, it is a useful trick to try, since it simplifies the calculations enough to give a tractable toy model calculation to practice with and to gain insight for the actual disk partition function calculation. So we consider a version of the series (6), where we replace the original integrals (8) by

$$\tilde{I}(N_1, N_2, N_3, \ldots; n_{\max}) = \int \left[\prod_{n=1}^{n_{\max}} \prod_{i=1}^{N_n} \frac{\mathrm{d}t_i^{(n)}}{2\pi}\right] \left[\prod_{n=1}^{n_{\max}} \prod_{1\leqslant i<j\leqslant N_n} \left|\mathrm{e}^{\mathrm{i}t_i^{(n)}} - \mathrm{e}^{\mathrm{i}t_j^{(n)}}\right|^{2n^2}\right]$$
$$\cdot \left[\prod_{1\leqslant n<m}^{n_{\max}} \prod_{i=1}^{N_n} \prod_{j=1}^{N_m} \left(1 - \frac{\mathrm{e}^{\mathrm{i}t_j^{(m)}}}{\mathrm{e}^{\mathrm{i}t_i^{(n)}}}\right)^{n^2} \left(1 - \frac{\mathrm{e}^{\mathrm{i}t_i^{(n)}}}{\mathrm{e}^{\mathrm{i}t_j^{(m)}}}\right)^{m^2}\right]$$
$$= \frac{\Gamma\left(1 + \sum_{n=1}^{n_{\max}} n^2 N_n\right)}{\prod_{n=1}^{n_{\max}} [\Gamma(1+n^2)]^{N_n}}, \tag{11}$$

where the last line is the exact analytical result for the integral [16, 17]. Since the integrals (11) are a variation of the Dyson gas integral formula (9), we call the new series 'Dyson series'. In appendix A we compare the original integrals I with the approximate ones $\tilde{I}$, for some cases where it is possible to calculate the original integral analytically, to see how much Dyson series toy model deviates from (11) the exact formula.

The virtue of the Dyson series is that we can also solve the task (ii): we can actually sum the series in a controlled way. We will first recognize it as an asymptotic series, but can rewrite it as an integral formula which we can regulate by a suitable deformation of integration contour. We will discuss that next.

4. Summing the Dyson series

Instead of the series (6) we consider the Dyson series with coefficients $\tilde{I}$ instead of I. We also simplified it further by truncating the infinite product, so that we have

$$Z_{\mathrm{Dyson}}(x^0; n_{\max}) = \left(\prod_{n=1}^{n_{\max}} \sum_{N_n=0}^{\infty} \frac{((-1)^n z_n)^{N_n}}{N_n!}\right) \frac{\Gamma\left(1 + \sum_{n=1}^{n_{\max}} n^2 N_n\right)}{\prod_{n=1}^{n_{\max}} [\Gamma(1+n^2)]^{N_n}}. \tag{12}$$

Even after truncating to a finite product of $n_{\max}$ terms, the expression is not well behaved since the product is that of possibly divergent infinite series. In order to gain better control, we rewrite (12) as an integral representation,

$$Z_{\mathrm{Dyson}}(x^0; n_{\max}) = \left(\prod_{n=1}^{n_{\max}} \sum_{N_n=0}^{\infty} \frac{((-1)^n z_n)^{N_n}}{N_n!(n^2)!^{N_n}}\right) \int_0^{\infty} \mathrm{d}u\, u^{\sum_{n=1}^{n_{\max}} n^2 N_n}\, \mathrm{e}^{-u}$$
$$= \int_0^{\infty} \mathrm{d}u \exp\left[-u + \sum_{n=1}^{n_{\max}} \frac{(-1)^n z_n u^{n^2}}{(n^2)!}\right]. \tag{13}$$

Now we have a single integral, and the exponent in the integrand is a finite sum of $n_{\max}$ terms. Let us take a closer look at it. We denote

$$F_{n_{\max}}(u) = -u + \sum_{n=1}^{n_{\max}} \frac{(-1)^n z_n u^{n^2}}{(n^2)!}. \tag{14}$$

For real u, $F_{n_{\max}}(u)$ is oscillatory with the amplitude of oscillation increasing with u. The largest oscillations are due to the terms with $n \simeq n_{\max}$. As a consequence, integral (13) does

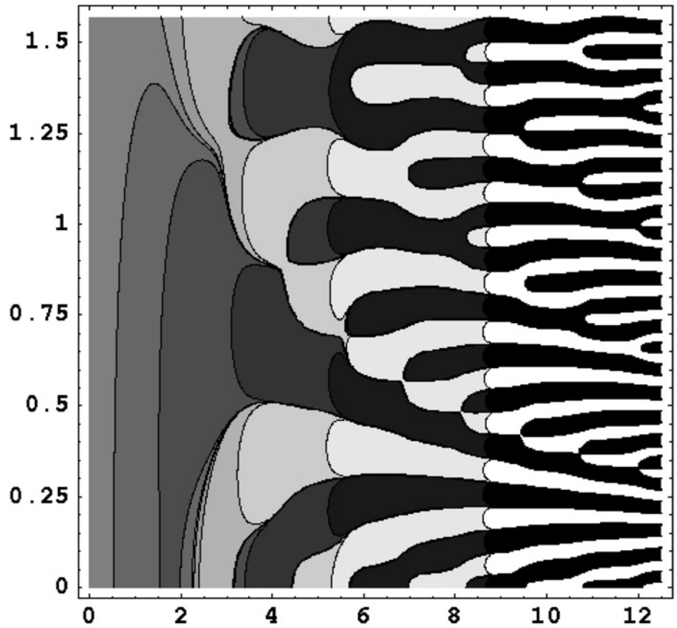

Figure 1. Re $F_{11}(r^2\,e^{i\phi})$ for $r = 0, \ldots, 12.5$ (horizontal axis) and for $\phi = 0, \ldots, \pi/2$ (vertical axis) with $x^0 = 0$. Re F_{11} is small in the dark regions.

not have expansion (12) for small z_n, and the limit $n_{\max} \to \infty$ does not exist. We will next give a prescription to regulate the integral.

Let us deform the contour of integration in (13) away from the positive real axis. If the integrand would be analytic, this would have no effect. However, it has an essential singularity at infinity. Consequently, the contour deformation will change the integral, due to a different approach to the point at infinity. Thus we can regulate the integral (13) by finding a suitable contour deformation. However, the integral will then also become complex valued. Since the pressure is real valued, we adopt a prescription where we define it to be the real part of the integral over the deformed contour[6],

$$Z_{\mathrm{Dyson}}(x^0; n_{\max}) = \mathrm{Re} \int_{\mathcal{C}} \mathrm{d}u \exp\left[F_{n_{\max}}(u)\right], \tag{15}$$

where $\mathcal{C}$ runs from 0 to ∞ such that $\mathrm{Re}F_{n_{\max}}$ decreases monotonically on it. For the choice of $\mathcal{C}$, see figure 1 which depicts the eye-appealing structure of the real part of $F_{n_{\max}}$ (the plot is shown for the value $n_{\max} = 11$). The regular structure of $\mathrm{Re}F_{n_{\max}}$ arises from the fact that $\mathrm{Re}F_{n_{\max}}(u)$ is dominated by the nth term of the sum at $|u| \simeq n^2$. Figure 1 suggests that there is a preferred choice for a path (in the quadrant $0 < \phi < \pi/2$) from 0 to ∞ that avoids all the light gray regions and proceeds in the direction of darker color (decreasing $\mathrm{Re}F_{n_{\max}}$). We call such a path $\mathcal{C}_{\mathrm{pref}}$ and focus on (15) with $\mathcal{C} = \mathcal{C}_{\mathrm{pref}}$ which stays well defined in the limit $n_{\max} \to \infty$.

[6] With this prescription, it reproduces the asymptotic series (12). If one has a strong preference to keep the integral real valued, one can alternatively first write it as a sum of two identical terms, then deform the contour in two opposite ways as mirror images of each other so that the two terms become complex conjugates.

As an example, let us consider the leading correction with $n_{\max} = 2$. We take $\mathcal{C}_{\text{pref}}$ with a constant phase, i.e., $u = r\,\mathrm{e}^{\mathrm{i}\pi/4}$ with $r = 0, \ldots, \infty$. Then

$$Z_{\text{Dyson}}(x^0; n_{\max} = 2) = \operatorname{Re} \int_0^{\infty} \mathrm{d}r \exp[\mathrm{i}\pi/4 - (1 + z_1) r\,\mathrm{e}^{\mathrm{i}\pi/4} - z_2 r^4/24], \tag{16}$$

which is well defined. (Recall that $z_n = z_n(x^0) \sim \exp(nx^0)$.) Developing the integrand at $z_2 = 0$, we get back the (asymptotic) series

$$Z_{\text{Dyson}}(x^0; 2) = \frac{1}{1 + z_1} + \frac{z_2}{(1 + z_1)^5} + \frac{35 z_2^2}{(1 + z_1)^9} + \cdots. \tag{17}$$

We want to compare this to the leading term

$$Z_{\text{Dyson}}(x^0; 1) \equiv \frac{1}{1 + z_1} = \frac{1}{1 + 2\pi\lambda\,\mathrm{e}^{x^0}}. \tag{18}$$

Numerical integration of (16) verifies that the *total* correction with $n_{\max} = 2$ is small,

$$\left[\frac{Z_{\text{Dyson}}(x^0; 2) - Z_{\text{Dyson}}(x^0; 1)}{Z_{\text{Dyson}}(x^0; 1)}\right]_{\max, x^0 \in R} \sim 10^{-3}, \tag{19}$$

and well described by the first few terms of the asymptotic series. Note then that at late times the first subleading term is $\sim z_2 z_1^{-5} \sim \mathrm{e}^{-3x^0}$, which is much smaller than the leading term $\sim z_1^{-1} \sim \mathrm{e}^{-x^0}$. One can argue that at late times $x^0 \to \infty$ all subleading terms are negligible compared to the leading e^{-x^0} behavior. Similarly, one finds that the $n_{\max} = 3$ correction is even smaller

$$\left[\frac{Z_{\text{Dyson}}(x^0; 3) - Z_{\text{Dyson}}(x^0; 2)}{Z_{\text{Dyson}}(x^0; 1)}\right]_{\max, x^0 \in R} \sim 10^{-7}. \tag{20}$$

Refining the approximation to larger values of $n_{\max}$ produces even more negligible corrections. Thus the total correction to the leading result (18) is at most $\sim 10^{-3}$ in the Dyson series, even when $n_{\max} \to \infty$. Thus, the approximate result for the disk partition function decays exponentially at late times,

$$Z_{\text{Dyson}}(x^0) = \lim_{n_{\max} \to \infty} Z_{\text{Dyson}}(x^0, n_{\max}) \sim \mathrm{e}^{-x^0}; \qquad x^0 \to \infty. \tag{21}$$

We will now return back to our original problem, the disk partition function (6). The lesson from the Dyson series toy model is that it is useful to truncate the infinite products by introducing a 'cut-off' $n_{\max}$ and then try to see how much the time dependence is corrected as $n_{\max}$ is increased. If the additional corrections are more and more subleading, they can be ignored in the limit $n_{\max} \to \infty$. The full series is, in fact, well approximated by just the leading terms as $x^0 \to \infty$. The partition function (6) turns out to have a similar behavior.

5. The original disk partition function at late times

Consider again the exact series (6). In our toy model the relevant late-time corrections are produced by the first terms in the asymptotic series (12). It turns out that the first terms of the exact series (6) can also be calculated analytically, without using any approximation for I. The first correction terms are those, where most of the $N_2, N_3, \ldots$ are zero. We denote the integral coefficients of these by

$$I_n(N_1, N_n) \equiv I(N_1, 0, 0, \ldots, 0, N_n, 0, 0, \ldots) \tag{22}$$

so, e.g., $I_2(N,4) = I(N_1 = N, N_2 = 4, 0, 0, \ldots)$. It turns out we can evaluate the integrals

$$I_n(N,1) = \int \frac{\mathrm{d}t_1^{(n)}}{2\pi} \prod_{i=1}^{N} \frac{\mathrm{d}t_i^{(1)}}{2\pi} \prod_{i<j} \left|\mathrm{e}^{it_i^{(1)}} - \mathrm{e}^{it_j^{(1)}}\right|^2 \prod_i \left|\mathrm{e}^{it_i^{(1)}} - \mathrm{e}^{it_1^{(n)}}\right|^{2n}. \tag{23}$$

This is a well-known Selberg integral, and has previously been applied in the context of rolling tachyons in [14]. The result reads

$$I_n(N,1) = N! \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2n)}{\Gamma(j+n)^2} = N! \prod_{j=0}^{n-1} \frac{j!}{(n+j)!} \frac{(N+n+j)!}{(N+j)!}. \tag{24}$$

In particular, we find

$$\begin{aligned} \frac{I_2(N,1)}{N!} &= \frac{N+2}{12} \frac{(N+3)!}{N!} = \binom{N+4}{4} + \binom{N+3}{4} \\ &= \frac{1}{4!}\left[\frac{(N+4)!}{N!} + \frac{(N+3)!}{(N-1)!}\right], \\ \frac{I_3(N,1)}{N!} &= \frac{1}{9!}\left[\frac{(N+9)!}{N!} + 10\frac{(N+8)!}{(N-1)!} + 20\frac{(N+7)!}{(N-2)!} + 10\frac{(N+6)!}{(N-3)!} + \frac{(N+5)!}{(N-4)!}\right], \end{aligned} \tag{25}$$

where the first terms of the sums are the same as in the Dyson series toy model.

Thus, we find the corrections to Z_{disk} (equation (6)) that are linear in $z_{2,3}$:

$$\begin{aligned} Z_{\mathrm{disk}}(x^0) &= \sum_{N=0}^{\infty} (-1)^N z_1^N \left[1 + z_2 \frac{I_2(N,1)}{N!} - z_3 \frac{I_3(N,1)}{N!} + \cdots\right] \\ &= \frac{1}{1+z_1} + \frac{z_2(1-z_1)}{(1+z_1)^5} - \frac{z_3\left(1 - 10z_1 + 20z_1^2 - 10z_1^3 + z_1^4\right)}{(1+z_1)^{10}} + \cdots. \end{aligned} \tag{26}$$

From (24) it follows that all higher order linear corrections (those depending on z_n with $n \geqslant 4$) have similar structures.

Note that the size of the corrections is slightly larger as in the Dyson series. In the latter, at late times the correction linear in z_2 was $\sim z_2 z_1^{-5} \sim \mathrm{e}^{-3x^0}$, but now we find $\sim z_2 z^{-4} \sim \mathrm{e}^{-2x^0}$. The correction linear in z_3 is subleading, we find at late times $\sim z_3 z_1^{-6} \sim \mathrm{e}^{-3x^0}$.

Moving to higher order, the coefficients $I_2(N,2)$ apparently also have a formula similar to (25). We find

$$\frac{I_2(N,2)}{2!N!} = \frac{1}{8!}\left[35\frac{(N+8)!}{N!} + 77\frac{(N+7)!}{(N-1)!} + 27\frac{(N+6)!}{(N-2)!} + \frac{(N+5)!}{(N-3)!}\right], \tag{27}$$

whence the correction to the disk partition function that is quadratic in z_2 becomes

$$\frac{z_2^2\left(35 - 77z_1 + 27z_1^2 - z_1^3\right)}{(1+z_1)^9}. \tag{28}$$

Interestingly, at late times this is of the same order as the linear correction, namely $\sim z_2^2 z_1^{-6} \sim \mathrm{e}^{-2x^0}$. As we will discuss below, at the order z_2^n we will similarly find $\sim z_2^n z_1^{-2n-2} \sim \mathrm{e}^{-2x^0}$, and generalizing to order z_3^n we will find $\sim z_3^n z_1^{-3n-3} \sim \mathrm{e}^{-3x^0}$. All these are small corrections compared to the leading $\sim \mathrm{e}^{-x^0}$ decay.

The above are still a tiny subset of all possible terms in the series (6), containing all possible combinations of monomials of $z_1, z_2, z_3, \ldots$. But we can estimate their late-time behavior too.

Equations (25), (27) show that the integers $I_n(N,1)$ and $I_2(N,2)$ can be expressed as finite sums over binomial coefficients. Using methods outlined in appendix B, we evaluated

$$\hat{I}(N_1, N_2, \ldots) = \frac{1}{\prod_n N_n!} I(N_1, N_2, \ldots) \tag{29}$$

for almost all fixed values of N_n for which $\hat{I} \lesssim 10^{19}$. Using these results we then discovered a generalizaton of formulae (25), (27) for more complicated sets of $N_2, N_3, \ldots$. We find that for any $N_1 = N$ with fixed $N_2, N_3, \ldots, N_{n_{\max}}$ (with $N_{n_{\max}} > 0$ and $0 = N_{n_{\max}+1} = N_{n_{\max}+2} = \cdots$), the $\hat{I}$ can be written as a finite sum

$$\begin{aligned}\hat{I}\big(N_1 = N, N_2, N_3, \ldots, N_{n_{\max}}\big) &= \frac{1}{S!}\sum_{\ell=0}^{\ell_{\max}} C_\ell \frac{(N+S-\ell)!}{(N-\ell)!} \\ &= \sum_{\ell=0}^{\ell_{\max}} C_\ell \binom{N+S-\ell}{S}\end{aligned} \tag{30}$$

where $S = \sum_{n=2}^{n_{\max}} n^2 N_n$. The relevant fact for the moment is that the coefficients C_ℓ turn out to be independent[7] of N. We will give an explicit formula for $\ell_{\max}$ below. The corresponding correction term to Z_{disk} then becomes

$$\begin{aligned}\delta Z_{\text{disk}} &= \prod_{n=2}^{n_{\max}} [(-1)^n z_n]^{N_n} \sum_{N=0}^{\infty} (-z_1)^N \hat{I}\big(N, N_2, N_3, \ldots, N_{n_{\max}}\big) \\ &= \prod_{n=2}^{n_{\max}} [(-1)^n z_n]^{N_n} \sum_{\ell=0}^{\ell_{\max}} C_\ell \sum_{N=0}^{\infty} \binom{N+S-\ell}{S} (-z_1)^N \\ &= \prod_{n=2}^{n_{\max}} [(-1)^n z_n]^{N_n} \sum_{\ell=0}^{\ell_{\max}} C_\ell \frac{(-z_1)^\ell}{(1+z_1)^{S+1}}.\end{aligned} \tag{31}$$

Importantly, for $\ell_{\max}$ we found[8] an explicit formula

$$\ell_{\max} = \sum_{n=2}^{n_{\max}} [n(n-1)N_n] - n_{\max} + 1. \tag{32}$$

The combination of (32) and the schematic formula (31) allows us to estimate the leading late-time dependence of all the correction terms to $Z_{\text{disk}}(x^0)$. At late times the leading part of the generic monomial correction (31) is given by the term with the highest exponent of z_1, i.e., the $\ell = \ell_{\max}$ term. Then, combining the late-time dependences

$$\begin{aligned}\prod_{n=2}^{n_{\max}} z_n^{N_n} &\sim \exp\left[\left(\sum_{n=2}^{n_{\max}} n N_n\right) x^0\right] \\ z_1^{\ell_{\max}} &\sim \exp\left[\left(\sum_{n=2}^{n_{\max}} n(n-1)N_n\right) x^0 - (n_{\max}-1)x^0\right] \\ z_1^{-(S+1)} &\sim \exp\left[-\left(\sum_{n=2}^{n_{\max}} n^2 N_n\right) - x^0\right],\end{aligned} \tag{33}$$

[7] Formula (30) has been evaluated and verified explicitly (with explicit coefficients C_ℓ), e.g., for $(N_2, N_3, N_4) = (1,1,0), (2,1,0), (0,2,0)$ and $(1,0,1)$ in addition to the cases discussed above.

[8] Using (24) it is straightforward to determine $\ell_{\max}$ for the corrections which are linear in z_n (with arbitrary $n = n_{\max}$). The general formula (32) was found by first making an educated guess and then testing it with computer calculations. So far we have explicitly verified it up to $n_{\max} = 4$, but have not yet been able to construct a general proof.

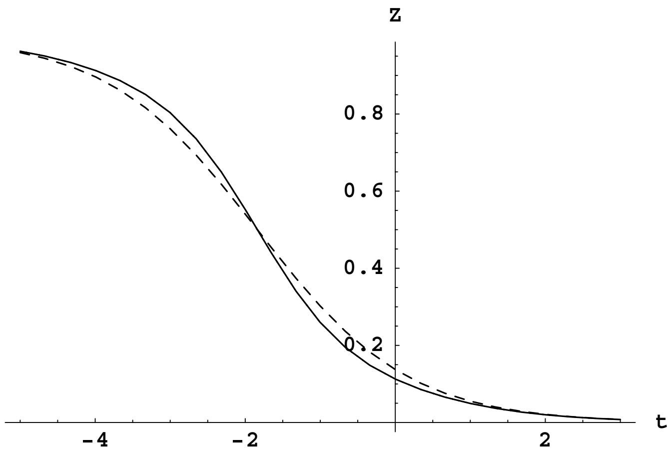

Figure 2. The disk partition function (35) as a function of time t. Here $\lambda = 1$ and we used a large value ~ 15 for β_2. For reference, the dashed line represents (3).

we find that the correction term (31) behaves as

$$\delta Z_{\text{disk}} \sim e^{-n_{\max} x^0} \tag{34}$$

at late times $x^0 \to +\infty$ and is thus subleading. Thus the leading correction is at most of the order e^{-2x^0}.

6. Summary

We have calculated the disk partition function with an oscillatory tachyon field profile (1) instead of the exactly marginal deformation (2). The largest deviations which we have found from the disk partition function (3) of the latter are surprisingly small, given by (26) and (28). Including the largest one (linear in z_2) the disk partition function reads

$$\begin{aligned} Z_{\text{disk}}(x^0) &\simeq \frac{1}{1 + 2\pi\lambda\, e^{x^0}} + \frac{z_2(1 - z_1)}{(1 + z_1)^5} \\ &= \frac{1}{1 + e^{\tilde{x}^0}} + \frac{\beta_2}{2\pi} \frac{\left(e^{2\tilde{x}^0} - e^{3\tilde{x}^0}\right)}{\left(1 + e^{\tilde{x}^0}\right)^5}, \end{aligned} \tag{35}$$

where $\tilde{x}^0 = x^0 + \ln 2\pi\lambda$. Figure 2 shows the disk partition function with $\lambda = 1$ and with a large value of $\beta_2 \simeq 15$ for better visualization. All the deviations seem to contribute around $x^0 = -\ln 2\pi\lambda$ and become smaller in size. We find the result surprising: the disk partition function is very similar to (3) although the tachyon profile (1) is oscillatory and very different from the monotonously rolling (2). In particular, the oscillatory behavior is almost washed out.

Acknowledgments

We thank Asad Naqvi for useful discussions and Barton Zwiebach for critical remarks on the earlier draft version of this work. NJ and MJ have been in part supported by the Magnus Ehrnrooth foundation. MJ also acknowledges a grant from GRASPANP, the Finnish Graduate

School in Particle and Nuclear Physics. This work was also partially supported by the EU 6th Framework Marie Curie Research and Training network 'UniverseNet' (MRTN-CT-2006-035863).

Appendix A. A simple method for evaluating *I*

Let us study an integral of the form

$$J_m = \int \prod_{i=1}^{m} \frac{\mathrm{d}t_i}{2\pi} \prod_{1\leqslant i<j\leqslant m} \left|\mathrm{e}^{\mathrm{i}t_i} - \mathrm{e}^{\mathrm{i}t_j}\right|^{2k_{ij}}, \tag{A.1}$$

where k_{ij} are integers. This form is a generalization of (8), where the exponents n^2 and nm are allowed to take any values. The integral may be expressed as a finite sum by doing a Fourier transform. The 'propagator' from t_i to t_j reads

$$S(t_j - t_i) = \left|1 - \mathrm{e}^{\mathrm{i}(t_j-t_i)}\right|^{2k_{ij}} = \sum_{n_{ij}=-k_{ij}}^{k_{ij}} (-1)^{n_{ij}} \binom{2k_{ij}}{k_{ij}+n_{ij}} \mathrm{e}^{in_{ij}(t_j-t_i)}. \tag{A.2}$$

By inserting this to (A.1) and by doing the t integrals we have

$$J_m = \left[\prod_{i<j}^{m} \sum_{n_{ij}=-k_{ij}}^{k_{ij}} (-1)^{n_{ij}} \binom{2k_{ij}}{k_{ij}+n_{ij}}\right] \prod_{i=1}^{m} \delta\left(\sum_{j=1}^{i-1} n_{ji} = \sum_{j=i+1}^{m} n_{ji}\right), \tag{A.3}$$

where only $m-1$ of the conditions in the (Kronecker) delta functions are independent. They can be used to fix the values of $n_{12}, n_{13}, \ldots$ so that

$$J_m = \left[\prod_{1<i<j\leqslant m} \sum_{n_{ij}=-k_{ij}}^{k_{ij}} (-1)^{n_{ij}} \binom{2k_{ij}}{k_{ij}+n_{ij}}\right] \prod_{j=2}^{m} \binom{2k_{1j}}{k_{1j} - \sum_{i=2}^{j-1} n_{ij} + \sum_{i=j+1}^{m} n_{ji}}. \tag{A.4}$$

This formula can be used to evaluate I for small n and N_n. E.g., $I_2(2,2)$ is found by letting $m = 4, k_{12} = 1, k_{13} = k_{14} = k_{23} = k_{24} = 2, k_{34} = 4$. Some values are tabulated in table A1. Note that

$$\tilde{I}_2(N_1, N_2) = \frac{(N_1 + 4N_2)!}{4!^{N_2}} \leqslant I(N_1, N_2). \tag{A.5}$$

In appendix B we present a more efficient method of evaluating I.

Appendix B. A formula for *I* using matrix determinants

In this appendix the integral

$$I(N_1, N_2, N_3 \ldots) = \int \left[\prod_{n=1}^{\infty}\prod_{i=1}^{N_n} \frac{\mathrm{d}t_i^{(n)}}{2\pi}\right] \left[\prod_{n=1}^{\infty} \prod_{1\leqslant i<j\leqslant N_n} \left|\mathrm{e}^{\mathrm{i}t_i^{(n)}} - \mathrm{e}^{\mathrm{i}t_j^{(n)}}\right|^{2n^2}\right] \cdot \left[\prod_{1\leqslant n<m}^{\infty} \prod_{i=1}^{N_n}\prod_{j=1}^{N_m} \left|\mathrm{e}^{\mathrm{i}t_i^{(n)}} - \mathrm{e}^{\mathrm{i}t_j^{(m)}}\right|^{2nm}\right] \tag{B.1}$$

is transformed to a finite sum over certain integer valued functions. This sum can then be used to evaluate I exactly for a given set of $\{N_n\}$.

Table A1. Comparison of I_2 and $\tilde{I}_2$.

N_1	N_2	$I_2(N_1, N_2)$	$\tilde{I}_2(N_1, N_2)$	point of interest
0	0	1	1	
k	0	$k!$	$k!$	
0	k	$\frac{(4k)!}{4!^k}$	$\frac{(4k)!}{4!^k}$	
1	1	$\frac{4!}{2!^2} = 6$	5	
2	1	$\frac{5!}{3} = \frac{5!3!}{2!} = 40$	30	
3	1	$5^3 \cdot 3 \cdot 2^2 = \frac{5 \cdot 5!}{2} = \frac{5 \cdot 6!}{3 \cdot 2^2} = 300$	210	
4	1	$2^3 3^2 \cdot 5 \cdot 7 = 7! = 2520$	1680	
1	2	$7^2 2^4 = \frac{7!^2 2!^4}{5!^2 3!^2} = 784$	630	$I(1, 2) = \left[\binom{9}{4} + \frac{14}{3}\right] I(1, 1)$
2	2	$5 \cdot 3^3 \cdot 17 \cdot 2^2 = 9180$	6300	
3	2	$2^4 \cdot 3 \cdot 2371 = 113\,808$	69 300	
1	3	$3 \cdot 2^5 7^2 11^2 = 569\,184$	450 450	$I(1, 3) = \left[\binom{13}{4} + 11\right] I(1, 2)$
2	3	$2^6 3^3 7 \cdot 13 \cdot 61 = 9592\,128$	6306 300	
1	4	$2^{10} 3^2 5^2 7^2 11^2 = 1366\,041\,600$	1072 071 000	$I(1, 4) = \left[\binom{17}{4} + 2^2 5\right] I(1, 3)$

For $n = 1$ (i.e., $0 = N_2 = N_3 = \cdots$), (B.1) becomes

$$I_N = \int \prod_i \frac{\mathrm{d}t_i}{2\pi} \prod_{1 \leqslant i < j \leqslant N} \left|\mathrm{e}^{\mathrm{i}t_i} - \mathrm{e}^{\mathrm{i}t_j}\right|^2. \tag{B.2}$$

Here the integrand is the absolute value squared of the Vandermonde determinant

$$\begin{aligned} |\Delta(z_1, \ldots, z_N)|^2 &= \prod_{1 \leqslant i < j \leqslant N} \left|\mathrm{e}^{\mathrm{i}t_i} - \mathrm{e}^{\mathrm{i}t_j}\right|^2 = \left| \sum_{\{i\}} \varepsilon_{i_1 \cdots i_N} z_1^{i_1 - 1} \cdots z_N^{i_N - 1} \right|^2 \\ &= \left| \sum_{\Pi} (-1)^{\Pi} \prod_{k=1}^{N} z_k^{\Pi(k) - 1} \right|^2 \end{aligned} \tag{B.3}$$

where $z_k = \exp(\mathrm{i}t_k)$ and Π denotes permutations of $1, 2, \ldots, N$. It is easy to check that if (B.3) is expressed as a polynomial of $\{z_k\}$, the constant term in the polynomial is equal to I_N.

The Vandermonde approach can be generalized for $n > 1$ using confluent Vandermonde matrices. This can be done by differentiation. For example,

$$\begin{aligned} \prod_{1 \leqslant i < j \leqslant N} |z_i - z_j|^{2 n_i n_j} &= \left| \frac{\partial}{\partial z_{N+1}} \Delta(z_1, \ldots, z_N, z_{N+1}) \Big|_{z_{N+1} = z_N} \right|^2 \\ &= \left| \sum_{\{i\}} \varepsilon_{i_1 \cdots i_{N+1}} z_1^{i_1 - 1} \cdots z_N^{i_N - 1} (i_{N+1} - 1) z_N^{i_{N+1} - 2} \right|^2, \end{aligned} \tag{B.4}$$

where $n_{I_N} = 2$ and all other $n_i = 1$. This is the determinant of a confluent Vandermonde matrix.

Generalizing to higher n and N_n (with $M = \sum_n n N_n < \infty$) the integrand in the definition of I becomes

$$\prod_{\text{pairs}} \left| z_i^{(n)} - z_j^{(m)} \right|^{2nm} = |\det A|^2, \tag{B.5}$$

where A is the $M \times M$ confluent Vandermonde matrix

$$A_{ij} = \frac{1}{(s-1)!} \left(\frac{\partial}{\partial z_k^{(n)}} \right)^{s-1} \left(z_k^{(n)} \right)^{j-1}. \tag{B.6}$$

The relation between n, k, s and i is (uniquely) determined by $1 \leqslant n$, $1 \leqslant k \leqslant N_n$, $1 \leqslant s \leqslant n$ and $\ell(n,k) + s = i$ with $\ell(n,k) = \sum_{m=1}^{n-1} m N_m + (k-1)n$. The result evaluates to

$$\begin{aligned}
\prod_{\text{pairs}} \left| z_i^{(n)} - z_j^{(m)} \right|^{2nm} &= \left| \sum_{\{i\}} \varepsilon_{i_1 \cdots i_M} \prod_n \prod_{k=1}^{N_n} \left[\prod_{s=1}^{n} \frac{1}{(s-1)!} \left(\frac{\partial}{\partial z_k^{(n)}} \right)^{s-1} \left(z_k^{(n)} \right)^{i_{\ell(n,k)+s}-1} \right] \right|^2 \\
&= \left| \sum_{\{i\}} \varepsilon_{i_1 \cdots i_M} \prod_n \prod_{k=1}^{N_n} \left[\prod_{s=1}^{n} \frac{(i_{\ell(n,k)+s} - 1) \cdots (i_{\ell(n,k)+s} - s + 1)}{(s-1)!} \left(z_k^{(n)} \right)^{i_{\ell(n,k)+s}-s} \right] \right|^2 \\
&= \left| \sum_{\{i\}} \varepsilon_{i_1 \cdots i_M} \prod_n \prod_{k=1}^{N_n} \frac{1}{n!(n-1)! \cdots 1!} \Delta(i_{\ell(n,k)+1}, \ldots, i_{\ell(n,k)+n}) \left(z_k^{(n)} \right)^{\sum_{s=1}^n i_{\ell(n,k)+s}} \right|^2
\end{aligned} \tag{B.7}$$

where $z_k^{(n)} = \exp\left(\mathrm{i} t_k^{(n)}\right)$, $\left| z_k^{(n)} \right|^2 = 1$ was used in the last step, and the Vandermonde matrices in the last form are obtained after antisymmetrization. Note that the complicated expression $\ell(n,k)$ is only needed for the pick up the permutation variable i with the correct index.

The constant term is

$$\begin{aligned}
I(N_1, N_2, \ldots) = \sum_{\{i\},\{j\}} \varepsilon_{i_1 \cdots i_M} \varepsilon_{j_1 \cdots j_M} \prod_n \prod_{k=1}^{N_n} \frac{1}{[n!(n-1)! \cdots 1!]^2} \\
\times \Delta(i_{\ell(n,k)+1}, \ldots, i_{\ell(n,k)+n}) \Delta(j_{\ell(n,k)+1}, \ldots, j_{\ell(n,k)+n}) \\
\times \delta(i_{\ell(n,k)+1} + \cdots + i_{\ell(n,k)+n}, j_{\ell(n,k)+1} + \cdots + j_{\ell(n,k)+n})
\end{aligned} \tag{B.8}$$

where $\delta(i,j) = \delta_{ij}$ is the Kronecker δ-symbol. For $n = 1$ the δ restrictions give simply $i_k = j_k$. Using these the result 'simplifies' to

$$\begin{aligned}
\frac{I(N_1, N_2, \ldots)}{N_1!} = \sum_{S,\{i\},\{j\}} \varepsilon_{i_1 \cdots i_K} \varepsilon_{j_1 \cdots j_K} \prod_{n>1} \prod_{k=1}^{N_n} \frac{1}{[n!(n-1)! \cdots 1!]^2} \\
\times \Delta(S(i_{\ell'(n,k)+1}), \ldots, S(i_{\ell'(n,k)+n})) \Delta(S(j_{\ell'(n,k)+1}), \ldots, S(j_{\ell'(n,k)+n})) \\
\times \delta\left(\sum_{s=1}^{n} S(i_{\ell'(n,k)+s}), \sum_{s=1}^{n} S(j_{\ell'(n,k)+s}) \right)
\end{aligned} \tag{B.9}$$

where $K = M - N_1$, the first sum goes over all increasing injections $S : \{1, \ldots, K\} \to \{1, \ldots, M\}$ (so that $i < j \Leftrightarrow S(i) < S(j)$), and $\ell'(n,k) = \ell(n,k) - N_1$.

Due to symmetry, one can add the restrictions $i_{\ell'(n,k)+1} < i_{\ell'(n,k+1)+1}$ (for all $n > 1$ and $1 \leqslant k < N_n$), and $i_{\ell'(n,k)+s} < i_{\ell'(n,k)+s+1}$, $j_{\ell'(n,k)+s} < j_{\ell'(n,k)+s+1}$ (for all $n > 1, k$, and $1 \leqslant s < n$) and multiply by the ratio of numbers of terms whence the result becomes

$$\begin{aligned}
\hat{I}(N_1, N_2, \ldots) &= \frac{I(N_1, N_2, \ldots)}{\prod_n N_n!} \\
&= {\sum_{S,\{i\},\{j\}}}' \varepsilon_{i_1 \cdots i_K} \varepsilon_{j_1 \cdots j_K} \prod_{n>1} \prod_{k=1}^{N_n} \frac{1}{[(n-1)! \cdots 1!]^2} \\
&\quad \times \Delta(S(i_{\ell'(n,k)+1}), \ldots, S(i_{\ell'(n,k)+n})) \Delta(S(j_{\ell'(n,k)+1}), \ldots, S(j_{\ell'(n,k)+n}))
\end{aligned}$$

$$\times\,\delta\left(\sum_{s=1}^{n} S(i_{\ell'(n,k)+s}),\ \sum_{s=1}^{n} S(j_{\ell'(n,k)+s})\right) \tag{B.10}$$

where the prime indicates the presence of the above restrictions. In particular,

$$\begin{aligned}\hat{I}_2(N_1, N_2) &= \hat{I}(N_1, N_2, 0, 0, \ldots)\\ &= \sum_{S,\{i\},\{j\}}{}' \varepsilon_{i_1\cdots i_K}\varepsilon_{j_1\cdots j_K}\prod_{k=1}^{N_2}(S(i_{2k-1}) - S(i_{2k}))(S(j_{2k-1}) - S(j_{2k}))\\ &\quad\times\delta(S(i_{2k-1}) + S(i_{2k}), S(j_{2k-1}) + S(j_{2k}))\end{aligned} \tag{B.11}$$

where $K = 2N_2$ and $\ell'(2, k) = 2(k-1)$ was inserted. We have written computer codes which evaluate I using formulae (B.10), (B.11) for a given (but arbitrary) set of $\{N_n\}$.

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